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Strong Approximation by Fourier Transforms and Fourier Series in L^∞-Norm*

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Let f be a complex-valued function belonging to $L^p(\mathbf{R})$ for some $1 < p < \infty$. We study the strong approximation of f, in $L^\infty(\mathbf{R})$-norm, by its Dirichlet integral, which is closely related to the Fourier transform of f. We prove sufficient conditions for f to belong to the saturation class $\mathcal{S}_p(\mathbf{R})$ in the case $2 \leqslant p < \infty$, and necessary conditions for f to belong to $\mathcal{S}_p(\mathbf{R})$ in the case $1 < p \leqslant 2$. As a consequence, we obtain a characterization of $\mathcal{S}_2(\mathbf{R})$. We formulate a conjecture on the characterization of $\mathcal{S}_p(\mathbf{R})$ in the case $1 < p < 2$, which is supported by our results on the strong approximation by Riesz means. Our machinery is also appropriate to prove sufficient or/and necessary conditions for the saturation class in connection with the strong approximation of a periodic function by the partial sum or Fejér mean of its Fourier series.

1. INTRODUCTION

We recall that the *Fourier transform* $\hat{f}$ of a complex-valued function $f \in L^1(\mathbf{R})$ is defined by

$$\hat{f}(u) := \frac{1}{2\pi} \int_{\mathbf{R}} f(x) e^{-iux}\, dx, \qquad u \in \mathbf{R}. \tag{1.1}$$

This definition makes sense in the case where $f \in L^p(\mathbf{R})$ for some $1 < p \leqslant 2$ if $\hat{f}(u)$ is defined as the limit of the truncated integrals

$$\frac{1}{2\pi} \int_{-v_1}^{v_2} f(x) e^{-iux}\, dx \qquad \text{as} \quad v_1, v_2 \to \infty$$

in $L^q(\mathbf{R})$-norm, where q denotes the *conjugate exponent* to $p : 1/p + 1/q = 1$. (See, e.g. [5, p. 96].)

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We note that the *inversion formula*

$$f(x) = \int_{\mathbf{R}} \hat{f}(u) e^{ixu}\, du =: 2\pi(\hat{f})^{\wedge}(-x), \qquad x \in \mathbf{R}, \tag{1.2}$$

also holds, where the integral in (1.2) is meant to be the limit of $\int_{-v_1}^{v_2}$ as $v_1, v_2 \to \infty$ in $L^p(\mathbf{R})$-norm.

Motivated by (1.2), the *Dirichlet integral* of a function $f \in L^p(\mathbf{R})$ for some $1 \leqslant p \leqslant 2$ is defined by

$$s_v(f, x) := \int_{-v}^{v} \hat{f}(u) e^{ixu}\, du, \tag{1.3}$$

furthermore, the *conjugate Dirichlet integral* is defined by

$$\tilde{s}_v(f, x) := \int_{-v}^{v} (-i \operatorname{sign} u) \hat{f}(u) e^{ixu}\, du, \qquad v \in \mathbf{R}_+, \quad x \in \mathbf{R}. \tag{1.4}$$

By (1.1) and Fubini's theorem, definitions (1.3) and (1.4) may be rewritten as follows

$$s_v(f, x) = \frac{1}{\pi} \int_{\mathbf{R}} f(x-t) \frac{\sin vt}{t}\, dt, \tag{1.5}$$

which justifies the use of the term "Dirichlet integral" as well, and

$$\tilde{s}_v(f, x) = \frac{1}{\pi} \int_{\mathbf{R}} f(x-t) \frac{1 - \cos vt}{t}\, dt. \tag{1.6}$$

The right-hand sides in (1.5) and (1.6) make sense even if $f \in L^p(\mathbf{R})$ for some $2 < p < \infty$, since these integrals exist in Lebesgue's sense, thanks to Hölder's inequality. In this paper, we shall use (1.5) and (1.6) in the capacity of the definitions of $s_v(f, x)$ and $\tilde{s}_v(f, x)$ for functions $f \in L^p(\mathbf{R})$ for some $2 < p < \infty$. We note that in this case the Fourier transform $\hat{f}(u)$ occurring in (1.3) and (1.4) exists only in the distributional sense in general.

We recall that the *Riesz mean* (*of first order*) of a function $f \in L^p(\mathbf{R})$ for some $1 \leqslant p < \infty$ is defined by

$$\sigma_v(f, x) := \frac{1}{v} \int_0^v s_\mu(f, x)\, d\mu, \tag{1.7}$$

while the *conjugate Riesz mean* is defined by

$$\tilde{\sigma}_\nu(f, x) := \frac{1}{\nu} \int_0^\nu \tilde{s}_\mu(f, x)\, d\mu, \qquad \nu \in \mathbf{R}_+, \quad x \in \mathbf{R}, \tag{1.8}$$

where $s_\mu(f, x)$ and $\tilde{s}_\mu(f, x)$ are defined in (1.5) and (1.6), respectively. By Fubini's theorem, we may write

$$\sigma_\nu(f, x) = \frac{1}{\pi} \int_{\mathbf{R}} f(x-t)\, \frac{1-\cos \nu t}{\nu t^2}\, dt, \tag{1.9}$$

$$\tilde{\sigma}_\nu(f, x) = \frac{1}{\pi} \int_{\mathbf{R}} f(x-t) \left(\frac{1}{t} - \frac{\sin \nu t}{\nu t^2}\right) dt. \tag{1.10}$$

We note that in the case where $f \in L^p(\mathbf{R})$ for some $1 \leqslant p \leqslant 2$, we may equally use definitions (1.3) and (1.4), respectively, which result in the following:

$$\sigma_\nu(f, x) = \int_{-\nu}^{\nu} \left(1 - \frac{|u|}{\nu}\right) \hat{f}(u) e^{ixu}\, du,$$

$$\tilde{\sigma}_\nu(f, x) = \int_{-\nu}^{\nu} \left(1 - \frac{|u|}{\nu}\right) (-i \operatorname{sign} u)\, \hat{f}(u) e^{ixu}\, du.$$

This is the reason why $\sigma_\nu(f, x)$ is also called the Cesàro mean of f. Furthermore, the right-hand side in (1.9) is well defined even for a function $f \in L^\infty(\mathbf{R})$, since the kernel $\varphi(u) := (1-\cos u)/\pi u^2$ belongs to $L^1(\mathbf{R})$.

It is known (see, e.g., [5, pp. 29–30]) that if $f \in L^p(\mathbf{R})$ for some $1 \leqslant p \leqslant \infty$, then

$$\sigma_\nu(f, x) - f(x) = \frac{1}{\nu} \int_0^\nu \{s_\mu(f, x) - f(x)\}\, d\mu \to 0 \qquad \text{as} \quad \nu \to \infty \tag{1.11}$$

for almost all $x \in \mathbf{R}$ (cf. (1.7)).

Next, we remind the reader that the *Hilbert transform* $\tilde{f}$ of a function $f \in L^p(\mathbf{R})$ for some $1 \leqslant p < \infty$ is defined in the principal value sense as follows

$$\tilde{f}(x) := (\text{P.V.})\, \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(x-t)}{t}\, dt$$

$$= -\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_\varepsilon^\infty \frac{f(x+t) - f(x-t)}{t}\, dt, \tag{1.12}$$

which exists for almost all $x \in \mathbf{R}$. By M. Riesz' theorem (see, e.g., [5, pp. 132–133]), for each $1 < p < \infty$ there exists a positive constant C_p such that

$$C_p^{-1} \|f\|_p \leqslant \|\tilde{f}\|_p \leqslant C_p \|f\|_p, \tag{1.13}$$

where

$$\|f\|_p := \left\{ \int_{\mathbf{R}} |f(x)|^p \, dx \right\}^{1/p}, \qquad f \in L^p(\mathbf{R}).$$

It is well known that if $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$, then for almost all $x \in \mathbf{R}$ we have

$$(\tilde{f})^{\sim}(x) := (\text{P.V.}) \frac{1}{\pi} \int_{\mathbf{R}} \frac{\tilde{f}(x-t)}{t} \, dt = -f(x) \tag{1.14}$$

and

$$\tilde{s}_\nu(f, x) = s_\nu(\tilde{f}, x), \qquad \tilde{\sigma}_\nu(f, x) = \sigma_\nu(\tilde{f}, x), \ \nu \in \mathbf{R}_+. \tag{1.15}$$

2. Strong Approximation by Dirichlet Integral

Motivated by the limit relation (1.11), the *strong approximation* of a function $f \in L^p(\mathbf{R})$ for some $1 \leqslant p < \infty$, in $L^\infty(\mathbf{R})$-norm, *by the Dirichlet integral* $s_\nu(f, x)$ is defined as follows

$$d_\nu(f, p) := \left\| \left\{ \frac{1}{\nu} \int_0^\nu |s_\mu(f, \cdot) - f(\cdot)|^p \, d\mu \right\}^{1/p} \right\|_\infty, \qquad \nu \in \mathbf{R}_+, \tag{2.1}$$

where

$$\|f\|_\infty := \operatorname{ess\,sup}\{ |f(x)| : x \in \mathbf{R} \}.$$

By Hölder's inequality, for $0 < p_1 < p_2 < \infty$ we have

$$\int_0^\nu |s_\mu(f, x) - f(x)|^{p_1} \, d\mu \leqslant \left\{ \int_0^\nu |s_\mu(f, x) - f(x)|^{p_2} \, d\mu \right\}^{p_1/p_2} \nu^{(p_2 - p_1)/p_2},$$

whence

$$d_\nu(f, p_1) \leqslant d_\nu(f, p_2), \qquad \nu \in \mathbf{R}_+. \tag{2.2}$$

We claim that the *saturation order* for $d_\nu(f, p)$ is $\nu^{-1/p}$. Indeed, if

$$d_\nu(f, p) = o(\nu^{-1/p}) \qquad \text{as} \quad \nu \to \infty,$$

then

$$\int_0^\infty |s_\nu(f, x) - f(x)|^p \, d\nu = 0$$

for almost all $x \in \mathbf{R}$. This implies that $f(x) = 0$ for almost all $x \in \mathbf{R}$.

We define the *saturation class* $\mathscr{S}_p(\mathbf{R})$ as follows

$$\mathscr{S}_p(\mathbf{R}) := \{ f \in L^p(\mathbf{R}) : d_\nu(f, p) = \mathcal{O}(\nu^{-1/p}) \text{ as } \nu \to \infty \}.$$

It is plain that a function $f \in L^p(\mathbf{R})$ for some $1 \leqslant p < \infty$ belongs to $\mathscr{S}_p(\mathbf{R})$ if and only if

$$\int_0^\infty |s_\nu(f, \cdot) - f(\cdot)|^p \, d\nu \in L^\infty(\mathbf{R}). \tag{2.3}$$

It is easy to see that $\mathscr{S}_p \cap L^\infty(\mathbf{R})$ is a *closed subspace* of $L^p \cap L^\infty(\mathbf{R})$. Indeed, we have

$$d_\nu(f_1 + f_2, p) \leqslant d_\nu(f_1, p) + d_\nu(f_2, p)$$

and

$$d_\nu(f, p) \leqslant C_p \, \|f\|_\infty, \qquad f_1, f_2, f \in L^p \cap L^\infty(\mathbf{R}). \tag{2.4}$$

Here and in the sequel, we denote by C_p a positive constant depending only on p, whose value may be different at different occurrences. Inequality (2.4) was proved in [1].

We shall prove the following

THEOREM 1. *If $f \in \mathscr{S}_p \cap L^\infty(\mathbf{R})$ for some $1 < p < \infty$, then*

$$f \in \operatorname{Lip} 1/p(\mathbf{R}) \tag{2.5}$$

and

$$\lim_{t \to 0} \frac{f(x+t) - f(x)}{|t|^{1/p}} = 0 \tag{2.6}$$

for almost all $x \in \mathbf{R}$.

Proof. (i) Let

$$V_\nu(f, x) := \frac{1}{\nu} \int_\nu^{2\nu} s_\mu(f, x)\, d\mu, \qquad \nu \in \mathbf{R}_+, \tag{2.6}$$

be the *de la Vallée Poussin mean* of f. By Hölder's inequality and (2.3), we get

$$\begin{aligned} |V_\nu(f, x) - f(x)| &\leqslant \frac{1}{\nu} \int_\nu^{2\nu} |s_\mu(f, x) - f(x)|\, d\mu \\ &\leqslant \frac{1}{\nu} \nu^{1-(1/p)} \left\{ \int_\nu^{2\nu} |s_\mu(f, x) - f(x)|^p\, d\mu \right\}^{1/p} \\ &= \mathcal{O}(\nu^{-1/p}) \end{aligned} \tag{2.7}$$

for almost all $x \in \mathbf{R}$. Now, [1, Theorem 1] implies (2.5).

(ii) The proof of (2.6) closely follows that of [3, Theorem 2]. Therefore, we only sketch it. Fix $\delta > 0$ and $K > 0$. By Egorov's theorem, there exists a measurable subset $\mathbf{P}_\delta$ of the interval $[-K, K]$ such that the Lebesgue measure of $[-K, K] \backslash \mathbf{P}_\delta$ is less than δ and the integrals

$$\int_0^\infty |s_\mu(f, x) - f(x)|^p\, d\mu$$

converge uniformly for each x in $\mathbf{P}_\delta$. Given $\varepsilon > 0$, there exists $\nu_0 > 0$ such that

$$\int_{\nu_0}^\infty |s_\mu(f, x) - f(x)|^p\, d\mu < \varepsilon^p, \qquad x \in \mathbf{P}_\delta.$$

Similarly to (2.7), hence we get

$$|V_\nu(f, x) - f(x)| < \varepsilon \nu^{-1/p}, \qquad \nu \geqslant \nu_0, \quad x \in \mathbf{P}_\delta.$$

Analyzing the proof of [1, Theorem 1] gives that

$$|f(x+t) - f(x)| \leqslant \varepsilon\, |t|^{1/p}, \qquad x \in \mathbf{P}_\delta,$$

for all sufficiently small $|t|$, $t \in \mathbf{R}$. Since δ, ε, and K are arbitrary positive numbers, we conclude (2.6) for almost all $x \in \mathbf{R}$.

3. Sufficient or Necessary Conditions for $f \in \mathscr{S}_p(\mathbf{R})$ via Hausdorff–Young Inequality

In this Section, we shall prove two theorems and three corollaries.

THEOREM 2. *Assume $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$ and $1/p + 1/q = 1$.*

(i) *If $2 \leqslant p < \infty$ and*

$$\int_0^\infty \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right|^q dt \in L^\infty(\mathbf{R}) \qquad \text{in } x, \tag{3.1}$$

then $f \in \mathscr{S}_p(\mathbf{R})$.

(ii) *If $1 < p \leqslant 2$ and $f \in \mathscr{S}_p(\mathbf{R})$, then (3.1) is satisfied.*

Observe that cases (i) and (ii) coincide in case $p = 2$.

COROLLARY 1. *If $f \in L^2(\mathbf{R})$, then $f \in \mathscr{S}_2(\mathbf{R})$ if and only if condition (3.1) is satisfied for $q = 2$.*

Proof of Theorem 2. By (1.5),

$$s_v(f, x) - f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \{f(x+t) + f(x-t) - 2f(x)\} \frac{\sin vt}{t} \, dt. \tag{3.2}$$

Let

$$F_x(t) := \frac{f(x+t) + f(x-t) - 2f(x)}{t}, \qquad t \in \mathbf{R}, \tag{3.3}$$

which is an odd function. Thus, (3.2) can be interpreted in such a way that the Fourier transform $\hat{F}_x$ of F_x is given by

$$\hat{F}_x(v) = -i\{s_v(f, x) - f(x)\}, \qquad v \in \mathbf{R}_+, \quad x \in \mathbf{R}. \tag{3.4}$$

(i) By the Hausdorff–Young inequality (see, e.g., [5, p. 96] or [6, Vol. 2, p. 254]) if $F_x(t) \in L^q(\mathbf{R})$ in t for some $1 < q \leqslant 2$, then $\hat{F}_x(v) \in L^p(\mathbf{R})$ in v and we have

$$\|\hat{F}_x(v)\|_p \leqslant C_p \, \|F_x(t)\|_q, \tag{3.5}$$

where the first norm is taken with respect to v, while the second norm with respect to t. In other words, we have

$$\left\{2\int_0^\infty |s_v(f,x)-f(x)|^p\,dv\right\}^{1/p}$$
$$\leqslant C_p\left\{\int_{\mathbf{R}}\left|\frac{f(x+t)+f(x-t)-2f(x)}{t}\right|^q dt\right\}^{1/q}$$

for all $x\in\mathbf{R}$. From (3.1) it follows that $f\in\mathcal{S}_p(\mathbf{R})$ and Part (i) is proved.

(ii) Again by the Hausdorff–Young inequality and by (1.2), if $\hat{F}_x(v)\in L^p(\mathbf{R})$ in v for some $1<p\leqslant 2$, then $F_x(t)\in L^q(\mathbf{R})$ in t and we have

$$\|F_x(t)\|_q\leqslant C_p\,\|\hat{F}_x(v)\|_p. \tag{3.6}$$

Applying this in the case of (3.4) yields

$$\left\{\int_{\mathbf{R}}\left|\frac{f(x+t)+f(x-t)-2f(x)}{t}\right|^q dt\right\}^{1/q}\leqslant C_p\left\{2\int_0^\infty |s_v(f,x)-f(x)|^p\,dv\right\}^{1/p}.$$

Since this is true for all $x\in\mathbf{R}$, hence (3.1) follows and Part (ii) is proved.

The next theorem gives a sufficient or a necessary condition for $f\in\mathcal{S}_p(\mathbf{R})$ in terms of the Hilbert transform $\tilde{f}$.

THEOREM 3. *Assume $f\in L^p(\mathbf{R})$ for some $1<p<\infty$ and $1/p+1/q=1$.*

(i) *If $2\leqslant p<\infty$ and*

$$\int_0^\infty\left|\frac{\tilde{f}(x+t)-\tilde{f}(x-t)}{t}\right|^q dt\in L^\infty(\mathbf{R})\qquad \text{in } x, \tag{3.7}$$

then $f\in\mathcal{S}_p(\mathbf{R})$.

(ii) *If $1<p\leqslant 2$ and $f\in\mathcal{S}_p(\mathbf{R})$, then (3.7) is satisfied.*

Observe that cases (i) and (ii) coincide in case $p=2$.

COROLLARY 2. *If $f\in L^2(\mathbf{R})$, then $f\in\mathcal{S}_2(\mathbf{R})$ if and only if condition (3.7) is satisfied for $q=2$.*

Combining Corollaries 1 and 2 yields the following

COROLLARY 3. *If $f\in L^2(\mathbf{R})$, then conditions (3.1) and (3.7) are equivalent for $q=2$.*

As a preparation for the proof of Theorem 3, we present another representation for the left-hand side in (3.2), which is interesting in itself.

LEMMA 1. *If $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$, then*

$$s_v(f, x) - f(x) = -\frac{1}{2\pi} \int_{\mathbf{R}} \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \frac{\cos vt}{t} \, dt, \qquad v \in \mathbf{R}_+, \tag{3.8}$$

for almost all $x \in \mathbf{R}$.

Proof. By (1.14) and (1.15), it is enough to prove that

$$\tilde{s}_v(f, x) - \tilde{f}(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \{f(x+t) - f(x-t)\} \frac{\cos vt}{t} \, dt$$

for all $x \in \mathbf{R}$, at which $\tilde{f}(x)$ exists. But the latter equality immediately follows from (1.6) and (1.12).

Proof of Theorem 3. This time, let

$$G_x(t) := \frac{\tilde{f}(x+t) - \tilde{f}(x-t)}{t}, \qquad t \in \mathbf{R},$$

which is "essentially" an even function (i.e., apart from a set of measure zero in t for any fixed $x \in \mathbf{R}$), whose Fourier transform $\hat{G}_x$ is given by

$$\hat{G}_x(v) = f(x) - s_v(f, x), \qquad v \in \mathbf{R}_+, \quad x \in \mathbf{R},$$

thanks to Lemma 1.

(i) Applying (3.5) with $G_x(t)$ instead of $F_x(t)$ gives

$$\left\{2 \int_0^{\infty} |s_v(f, x) - f(x)|^p \, dv\right\}^{1/p} \leqslant C_p \left\{\int_{\mathbf{R}} \left|\frac{\tilde{f}(x+t) - \tilde{f}(x-t)}{t}\right|^q dt\right\}^{1/q}.$$

This holds for almost all $x \in \mathbf{R}$. By (3.7), we conclude $f \in \mathscr{S}_p(\mathbf{R})$.

(ii) Applying (3.6) again with $G_x(t)$ instead of $F_x(t)$ yields

$$\left\{\int_{\mathbf{R}} \left|\frac{\tilde{f}(x+t) - \tilde{f}(x-t)}{t}\right|^q dt\right\}^{1/q} \leqslant C_p \left\{2 \int_0^{\infty} |s_v(f, x) - f(x)|^p \, dv\right\}^{1/p}.$$

This is true for almost all $x \in \mathbf{R}$. Hence (3.7) follows.

On closing, we note that the analogues of Theorems 1–3 were proved by Freud [3] in the case of Fourier series on the torus $\mathbf{T} := [-\pi, \pi)$.

4. Sufficient or Necessary Conditions for $f \in \mathcal{S}_p(\mathbf{R})$ via Pitt's Inequality

Similarly to Theorems 2 and 3, we can deduce sufficient or necessary conditions for $f \in \mathcal{S}_p(\mathbf{R})$ if, instead of the Hausdorff–Young inequality, we make use of Pitt's inequality (see, e.g. [4, p. 569]). For the reader's convenience, we formulate it in the following

Lemma 2. *Assume $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$, $0 \leqslant b < 1$, and $a := b + p - 2 \geqslant 0$. If $|f(x)|^p\, |x|^a \in L^1(\mathbf{R})$, then $\hat{f}(u)$ exists in the sense of ordinary function and*

$$\int_{\mathbf{R}} |\hat{f}(u)|^p\, |u|^{-b}\, du \leqslant C_{b,p} \int_{\mathbf{R}} |f(x)|^p\, |x|^a\, dx. \tag{4.1}$$

The counterpart of Theorem 2 reads as follows.

Theorem 4. *Assume $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$.*

(i) *If $2 \leqslant p < \infty$ and*

$$\int_0^\infty \frac{|f(x+t) + f(x-t) - 2f(x)|^p}{t^2}\, dt \in L^\infty(\mathbf{R}) \qquad \text{in } x, \tag{4.2}$$

then $f \in \mathcal{S}_p(\mathbf{R})$.

(ii) *If $1 < p \leqslant 2$ and $f \in \mathcal{S}_p(\mathbf{R})$, then (4.2) is satisfied.*

We note that conditions (3.1) and (4.2) coincide for $p = q = 2$.

Proof of Theorem 4. (i) Making use of representation (3.2) (together with (3.3) and (3.4)), by (4.1) we have

$$\int_0^\infty |s_v(f, x) - f(x)|^p\, dv \leqslant C_{0,p} \int_0^\infty \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right|^p t^{p-2}\, dt$$

for all $x \in \mathbf{R}$. Hence, (4.2) implies $f \in \mathcal{S}_p(\mathbf{R})$.

(ii) By the inversion formula (1.2), in the case $1 < p \leqslant 2$ inequality (4.1) remains valid if we interchange the roles of $\hat{f}(u)$ and $f(x)$. As a result, we find

$$\int_0^\infty \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right|^p t^{p-2}\, dt \leqslant C_{2-p,p} \int_0^\infty |s_v(f, x) - f(x)|^p\, dv.$$

This holds for all $x \in \mathbf{R}$. Hence (4.2) follows.

The counterpart of Theorem 3 reads as follows.

THEOREM 5. *Assume* $f \in L^p(\mathbf{R})$ *for some* $1 < p < \infty$.

(i) *If* $2 \leqslant p < \infty$ *and*

$$\int_0^{\infty} \frac{|\tilde{f}(x+t)-\tilde{f}(x-t)|^p}{t^2}\, dt \in L^{\infty}(\mathbf{R}) \qquad \text{in } x, \tag{4.3}$$

then $f \in \mathscr{S}_p(\mathbf{R})$.

(ii) *If* $1 < p \leqslant 2$ *and* $f \in \mathscr{S}_p(\mathbf{R})$, *then* (4.3) *is satisfied.*

We note that conditions (3.7) and (4.3) coincide for $p = q = 2$.

Proof of Theorem 5. It is analogous to the proof of Theorem 4, with the exception that time we use representation (3.8) instead of (3.2). We omit the details.

5. Strong Approximation by Riesz Means

Similarly to (2.1), we may define the *strong approximation* of a function $f \in L^p(\mathbf{R})$ for some $1 \leqslant p < \infty$, in $L^{\infty}(\mathbf{R})$-norm, *by the Riesz mean* $\sigma_\nu(f, x)$ as follows

$$r_\nu(f, p) := \left\| \left\{ \frac{1}{\nu} \int_0^{\nu} |\sigma_\mu(f, \cdot) - f(\cdot)|^p \, d\mu \right\}^{1/p} \right\|_{\infty}, \qquad \nu \in \mathbf{R}_+. \tag{5.1}$$

Again, for $0 < p_1 < p_2 < \infty$ we have

$$r_\nu(f, p_1) \leqslant r_\nu(f, p_2)$$

(cf. (2.2)) and the *saturation order* for $r_\nu(f, p)$ is $\nu^{-1/p}$, which means that if

$$r_\nu(f, p) = o(\nu^{-1/p}) \qquad \text{as} \quad \nu \to \infty,$$

then $f(x) = 0$ for almost all $x \in \mathbf{R}$. Furthermore, the *saturation class* is defined by the condition

$$r_\nu(f, p) = \mathcal{O}(\nu^{-1/p}) \qquad \text{as} \quad \nu \to \infty,$$

which is equivalent to the requirement (cf. (2.3))

$$\int_0^{\infty} |\sigma_\nu(f, \cdot) - f(\cdot)|^p \, d\nu \in L^{\infty}(\mathbf{R}). \tag{5.2}$$

The novelty is that if we substitute $\sigma_\nu(f, x)$ for $s_\nu(f, x)$, then Part (i) in Theorems 4 and 5 can be extended for the missing case $1 < p \leqslant 2$, too.

THEOREM 4′. *Assume $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$. If condition* (4.2) *is satisfied, then relation* (5.2) *is also satisfied.*

THEOREM 5′. *Assume $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$. If condition* (4.3) *is satisfied, then relation* (5.2) *is also satisfied.*

These two theorems give rise to the following

Conjecture 1. For $1 < p < 2$, relation (2.3) is equivalent to each of the conditions (4.2) and (4.3).

By Corollaries 1 and 2, this is valid for $p = 2$.

If Conjecture 1 were true, then conditions (4.2) and (4.3) would be equivalent in the case of any function $f \in L^p(\mathbf{R})$ for some $1 < p \leqslant 2$, and the saturation class $\mathcal{S}_p(\mathbf{R})$ would be characterized by any of them.

Before proving Theorems 4′ and 5′, we reformulate three auxiliary results from [2] in the form of the following

LEMMA 3. *If $g \in L^p(\mathbf{R})$ for some $1 < p < \infty$, then*

$$\int_0^\infty \left| \frac{1}{v} \int_0^v g(t)\, dt \right|^p dv \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty |g(t)|^p\, dt, \tag{5.3}$$

$$\int_0^\infty \left| v \int_0^{1/v} g(t)\, dt \right|^p dv \leqslant \left(\frac{p}{p+1} \right)^p \int_0^\infty |g(t)|^p \frac{dt}{t^2}, \tag{5.4}$$

$$\int_0^\infty \left| \frac{1}{v} \int_{1/v}^\infty g(t) \frac{dt}{t^2} \right|^p dv \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty |g(t)|^p \frac{dt}{t^2}, \tag{5.5}$$

$$\int_0^\infty \left| \int_0^{1/v} g(t)\, dt \right|^p dv \leqslant p^p \int_0^\infty |tg(t)|^p \frac{dt}{t^2}. \tag{5.6}$$

As a hint, we refer to [2, Lemma 5] which gives (5.3), to [2, Lemma 6] which gives (5.4) and (5.5) in the particular case where $\alpha := 1/p$ and $r := p$, and to [2, Lemma 4] which gives (5.6) in the particular case where $\alpha := (p-1)/p$ and $r := p$.

Proof of Theorem 4′. We note that (5.2) for $2 \leqslant p < \infty$ is a consequence of Part (i) in Theorem 4. In fact, by the left-hand side equality in (1.11) and by (5.3), we estimate as follows

$$\left\{ \int_0^\infty |\sigma_v(f, x) - f(x)|^p\, dv \right\}^{1/p} = \left\{ \int_0^\infty \left| \frac{1}{v} \int_0^v \{ s_\mu(f, x) - f(x) \}\, d\mu \right|^p dv \right\}^{1/p}$$
$$\leqslant \frac{p}{p-1} \left\{ \int_0^\infty |s_v(f, x) - f(x)|^p\, dv \right\}^{1/p}. \tag{5.7}$$

This shows that (5.2) holds for $2 \leqslant p < \infty$. However, this statement is weaker than Part (i) in Theorem 4.

In the general case where $1 < p < \infty$, we may proceed as follows. By (1.9), we get the representation

$$\sigma_v(f, x) - f(x) = \frac{1}{\pi} \int_0^\infty \{f(x+t) + f(x-t) - 2f(x)\} \frac{1 - \cos vt}{vt^2} \, dt$$
$$=: \frac{1}{\pi} \left\{ \int_0^{1/v} + \int_{1/v}^\infty \right\}. \tag{5.8}$$

By Minkowski's inequality,

$$\left\{ \int_0^\infty |\sigma_v(f, x) - f(x)|^p \, dv \right\}^{1/p}$$
$$\leqslant \frac{1}{\pi} \left\{ \int_0^\infty \left| \int_0^{1/v} \{f(x+t) + f(x-t) - 2f(x)\} \frac{1 - \cos vt}{vt^2} \, dt \right|^p dv \right\}^{1/p}$$
$$+ \frac{1}{\pi} \left\{ \int_0^\infty \left| \int_{1/v}^\infty \{f(x+t) + f(x-t) - 2f(x)\} \frac{1 - \cos vt}{vt^2} \, dt \right|^p dv \right\}^{1/p}$$
$$=: \frac{1}{\pi} (I_1 + I_2), \qquad \text{say.} \tag{5.9}$$

Since

$$\frac{1 - \cos vt}{vt^2} \leqslant \min \left\{ \frac{v}{2}, \frac{2}{vt^2} \right\}, \qquad v, t \in \mathbf{R}_+,$$

by (5.4) and (5.5), we obtain

$$I_1 \leqslant \frac{1}{2} \left\{ \int_0^\infty \left(v \int_0^{1/v} |f(x+t) + f(x-t) - 2f(x)| \, dt \right)^p dv \right\}^{1/p}$$
$$\leqslant \frac{p}{2(p+1)} \left\{ \int_0^\infty \frac{|f(x+t) + f(x-t) - 2f(x)|^p}{t^2} \, dt \right\}^{1/p} \tag{5.10}$$

and

$$I_2 \leqslant 2 \left\{ \int_0^\infty \left(\frac{1}{v} \int_{1/v}^\infty |f(x+t) + f(x-t) - 2f(x)| \frac{dt}{t^2} \right)^p dv \right\}^{1/p}$$
$$\leqslant \frac{2p}{p-1} \left\{ \int_0^\infty \frac{|f(x+t) + f(x-t) - 2f(x)|^p}{t^2} \, dt \right\}^{1/p}. \tag{5.11}$$

Combining (5.9)–(5.11) yields (5.2) to be proved.

Beside representation (5.8), we need another one in terms of the Hilbert transform $\tilde{f}$ (cf. Lemma 1).

LEMMA 4. *If $f \in L^p(\mathbf{R})$ for some $1 < p < \infty$, then*

$$\sigma_\nu(f, x) - f(x) = -\frac{1}{\pi} \int_0^\infty \{\tilde{f}(x+t) - \tilde{f}(x-t)\} \frac{\sin \nu t}{\nu t^2}\, dt, \qquad \nu \in \mathbf{R}_+, \tag{5.12}$$

for almost all $x \in \mathbf{R}$.

Proof. By (1.14) and (1.15), it is enough to prove that

$$\tilde{\sigma}_\nu(f, x) - \tilde{f}(x) = -\frac{1}{\pi} \int_0^\infty \{f(x+t) - f(x-t)\} \frac{\sin \nu t}{\nu t^2}\, dt$$

for all $x \in \mathbf{R}$, at which $\tilde{f}(x)$ exists. But the latter equality immediately follows from (1.10) and (1.12).

Proof of Theorem 5′. By (5.12) and Minkowski's inequality, we get

$$\begin{aligned} &\left\{\int_0^\infty |\sigma_\nu(f, x) - f(x)|^p\, d\nu\right\}^{1/p} \\ &\leqslant \frac{1}{\pi} \left\{\int_0^\infty \left|\int_0^{1/\nu} \frac{\tilde{f}(x+t) - \tilde{f}(x-t)}{t} \frac{\sin \nu t}{\nu t}\, dt\right|^p d\nu\right\}^{1/p} \\ &\quad + \frac{1}{\pi} \left\{\int_0^\infty \left|\int_{1/\nu}^\infty \frac{\tilde{f}(x+t) - \tilde{f}(x-t)}{t} \frac{\sin \nu t}{\nu t}\, dt\right|^p d\nu\right\}^{1/p} \\ &=: \frac{1}{\pi}(I_3 + I_4), \qquad \text{say.} \end{aligned} \tag{5.13}$$

By (5.6), we have

$$\begin{aligned} I_3 &\leqslant \left\{\int_0^\infty \left(\int_0^{1/\nu} \frac{|\tilde{f}(x+t) - \tilde{f}(x-t)|}{t}\, dt\right)^p d\nu\right\}^{1/p} \\ &\leqslant p \left\{\int_0^\infty \frac{|\tilde{f}(x+t) - \tilde{f}(x-t)|^p}{t^2}\, dt\right\}^{1/p}, \end{aligned} \tag{5.14}$$

while by (5.5),

$$\begin{aligned} I_4 &\leqslant \left\{\int_0^\infty \left(\frac{1}{\nu} \int_{1/\nu}^\infty \frac{|\tilde{f}(x+t) - \tilde{f}(x-t)|}{t^2}\, dt\right)^p d\nu\right\}^{1/p} \\ &\leqslant \frac{p}{p-1} \left\{\int_0^\infty \frac{|\tilde{f}(x+t) - \tilde{f}(x-t)|}{t^2}\, dt\right\}^{1/p}. \end{aligned} \tag{5.15}$$

Combining (5.13)–(5.15) gives (5.2) to be proved.

6. Strong Approximation by Fourier Series

It is of some interest to point out that the analogues of Theorems 4, 5, 4′, and 5′ can also be proved for Fourier series on the torus $\mathbf{T} := [-\pi, \pi)$.

To go into details, we briefly recall that the *Fourier series*

$$\tfrac{1}{2} a_0(f) + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \tag{6.1}$$

of a function $f \in L^1(\mathbf{T})$ is defined by

$$a_k(f) := \frac{1}{2\pi} \int_{\mathbf{T}} f(x) \cos kx \, dx, \qquad b_k(f) := \frac{1}{2\pi} \int_{\mathbf{T}} f(x) \sin kx \, dx, \; k \in \mathbf{Z}_+.$$

Let

$$s_n(f, x) := \tfrac{1}{2} a_0(f) + \sum_{k=1}^{n} (a_k(f) \cos kx + b_k(f) \sin kx)$$

be the nth *partial sum* of (6.1), and let

$$\sigma_n(f, x) := \frac{1}{n+1} \sum_{k=0}^{n} s_k(f, x), \qquad n \in \mathbf{Z}_+,$$

be the nth *Fejér* (or *first arithmetic*) *mean* of (6.1).

It is well known that if $f \in L^1(\mathbf{T})$, then

$$\sigma_n(f, x) - f(x) = \frac{1}{n+1} \sum_{k=0}^{n} \{s_k(f, x) - f(x)\} \to 0 \qquad \text{as} \quad n \to \infty \tag{6.2}$$

for almost all $x \in \mathbf{T}$.

Next, we remind the reader that the *conjugate function* $\tilde{f}^*$ of $f \in L^1(\mathbf{T})$ is defined in the principal value sense as follows

$$\begin{aligned} \tilde{f}^*(x) &:= (\text{P.V.}) \frac{1}{\pi} \int_{\mathbf{T}} \frac{f(x-t)}{2 \tan(t/2)} \, dt \\ &= -\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\varepsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2 \tan(t/2)} \, dt \end{aligned}$$

(cf. (1.12)). It is well known that $\tilde{f}^*(x)$ exists for almost all $x \in \mathbf{T}$ and inequalities (1.13) hold for $\tilde{f}^*$ instead of $\tilde{f}$, where $f \in L^p(\mathbf{T})$ for some $1 < p < \infty$.

Motivated by the limit relation (6.2), the *strong approximation* of a function $f \in L^p(\mathbf{T})$ for some $1 \leqslant p < \infty$, in $L^\infty(\mathbf{T})$-norm, *by the partial sum* $s_n(f, x)$ is defined as follows

$$d_n^*(f, p) := \left\| \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s_k(f, \cdot) - f(\cdot)|^p \right\}^{1/p} \right\|_\infty, \qquad n \in \mathbf{Z}_+.$$

Again, the *saturation order* for $d_n^*(f, p)$ is $n^{-1/p}$, which means that if for some $1 < p < \infty$ we have

$$d_n^*(f, p) = o(n^{-1/p}) \qquad \text{as} \quad n \to \infty,$$

then $f(x) = \text{constant}$ for almost all $x \in \mathbf{T}$. Furthermore, the *saturation class* $\mathcal{S}_p(\mathbf{T})$ is defined by the condition

$$d_n^*(f, p) = \mathcal{O}(n^{-1/p}) \qquad \text{as} \quad n \to \infty,$$

which is equivalent to the requirement

$$\sum_{k=0}^{\infty} |s_k(f, \cdot) - f(\cdot)|^p \in L^\infty(\mathbf{T}). \tag{6.3}$$

Analogously to Theorems 4 and 5, one can prove the following two theorems.

THEOREM 6. *Assume $f \in L^p(\mathbf{T})$ for some $1 < p < \infty$.*

(i) *If $2 \leqslant p < \infty$ and*

$$\int_0^\pi \frac{|f(x+t) + f(x-t) - 2f(x)|^p}{t^2}\, dt \in L^\infty(\mathbf{T}) \qquad \text{in } x, \tag{6.4}$$

then relation (6.3) *is satisfied.*

(ii) *If $1 < p \leqslant 2$ and* (6.3) *is satisfied, then condition* (6.4) *is also satisfied.*

THEOREM 7. *Assume $f \in L^p(\mathbf{T})$ for some $1 < p < \infty$.*

(i) *If $2 \leqslant p < \infty$ and*

$$\int_0^\pi \frac{|\tilde{f}(x+t) - \tilde{f}(x-t)|^p}{t^2}\, dt \in L^\infty(\mathbf{T}) \qquad \text{in } x, \tag{6.5}$$

then relation (6.3) *is satisfied.*

(ii) *If* $1<p\leqslant 2$ *and* (6.3) *is satisfied, then condition* (6.5) *is also satisfied.*

Finally, we consider the *strong approximation* of a function $f\in L^p(\mathbf{T})$ for some $1\leqslant p<\infty$, in $L^{\infty}(\mathbf{T})$-norm, *by the Cesàro mean* $\sigma_n(f,x)$ defined as follows

$$r_n^*(f,p):=\left\|\left\{\frac{1}{n+1}\sum_{k=0}^{n}|\sigma_k(f,\cdot)-f(\cdot)|^p\right\}^{1/p}\right\|_{\infty}.$$

The *saturation order* for $r_n^*(f,p)$ is again $n^{-1/p}$ and the *saturation class* is defined by the condition

$$r_n^*(f,p)=\mathcal{O}(n^{-1/p})\qquad\text{as}\quad n\to\infty,$$

which is equivalent to the requirement

$$\sum_{k=0}^{\infty}|\sigma_k(f,\cdot)-f(\cdot)|^p\in L^{\infty}(\mathbf{T}). \tag{6.6}$$

Analogously to Theorems 4′ and 5′, one can prove the following two theorems.

THEOREM 6′. *Assume* $f\in L^p(\mathbf{T})$ *for some* $1<p<\infty$. *If condition* (6.4) *is satisfied, then relation* (6.6) *is also satisfied.*

THEOREM 7′. *Assume* $f\in L^p(\mathbf{T})$ *for some* $1<p<\infty$. *If condition* (6.5) *is satisfied, then relation* (6.6) *is also satisfied.*

Theorems 6′ and 7′ give rise to the following

Conjecture 2. For $1<p<2$, relation (6.6) is equivalent to each of the conditions (6.4) and (6.5).

This is certainly true for $p=2$ by the results of Freud [3, Theorems 3 and 4].

If Conjecture 2 were true, then conditions (6.4) and (6.5) would be equivalent in the case of any function $f\in L^p(\mathbf{T})$ for some $1<p\leqslant 2$, and the saturation class (i.e., the class of those functions $f\in L^p(\mathbf{T})$ for which relation (6.3) is satisfied) would be characterized by any of them.

REFERENCES

1. DANG VU GIANG, Approximation on the real line by Fourier transform, *Acta Sci. Math. (Szeged)* **58** (1993), 197–209.
2. DANG VU GIANG AND F. MÓRICZ, A new characterization of Besov spaces on the real line, *J. Math. Anal. Appl.* **189** (1995), 533–551.
3. G. FREUD, Über die Sättigungsklasse der starken Approximation durch Teilsummen der Fourierschen Reihe, *Acta Math. Acad. Sci. Hungar.* **20** (1969), 275–279.
4. J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, "Weighted Norm Inequalities and Related Topics," North-Holland, Amsterdam/New York/Oxford, 1985.
5. E. C. TITCHMARSH, "Introduction to the Theory of Fourier Integrals," Clarendon Press, Oxford, 1937.
6. A. ZYGMUND, "Trigonometric Series," Cambridge Univ. Press, Cambridge, UK, 1959.